

# On $h(x)$ – Fibonacci polynomials in an arbitrary algebra

Cristina Flaut, Vitalii Shpakivskyi and Elena Vlad

**Abstract.** In this paper, we introduce  $h(x)$  – Fibonacci polynomials in an arbitrary finite-dimensional unitary algebra over a field  $K$  ( $K = \mathbb{R}, \mathbb{C}$ ), which generalize both  $h(x)$  – Fibonacci quaternion polynomials and  $h(x)$  – Fibonacci octonion polynomials. For  $h(x)$  – Fibonacci polynomials in such an arbitrary algebra, we prove summation formula, generating function, Binet-style formula, Catalan-style identity, and d’Ocagne-type identity.

MSC 2010: Primary 11B39; Secondary 11R54

## Introduction

In modern investigation there is a huge interest to real Fibonacci numbers and numerous their generalizations. The Fibonacci numbers  $f_n$  are the terms of the sequence  $0, 1, 1, 2, 3, 5, \dots$ , where

$$f_n = f_{n-1} + f_{n-2}, \quad n = 2, 3, \dots,$$

with the initial values  $f_0 = 0, f_1 = 1$ .

In the paper [1], were introduced  $k$ –Fibonacci numbers  $f_{k,n}$  by the equality

$$f_{k,n} = kf_{k,n-1} + f_{k,n-2}, \quad n = 2, 3, \dots,$$

with the initial values  $f_{k,0} = 0, f_{k,1} = 1$ , for all  $k$ . It is easy to see that Pell numbers are 2–Fibonacci numbers. Catalan studied polynomials  $F_n(x)$ , defined by the recurrence relation

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad n \geq 3,$$

where  $F_1(x) = 0, F_2(x) = x$ . There are another generalizations of Fibonacci numbers, as for example, Jacobsthal polynomials  $J_n(x)$ , defined by

$$J_n(x) = J_{n-1}(x) + xJ_{n-2}(x), \quad n = 3, 4, \dots,$$

where  $J_1(x) = J_2(x) = 1$ . There also exist the Byrd polynomials  $\varphi_n(x)$  which are defined by the relation

$$\varphi_n(x) = 2x\varphi_{n-1}(x) + \varphi_{n-2}(x), \quad n = 2, 3, \dots,$$

where  $J_0(x) = 0, \varphi_1(x) = 1$ . The Lucas polynomials  $L_n(x)$  were introduction by Bicknell, and are defined by

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x), \quad n = 2, 3, \dots,$$

where  $L_0(x) = 2, L_1(x) = x$ .

In the paper [4], authors introduced  $h(x)$ –Fibonacci polynomials  $F_{h,n}(x)$ , which generalize both Catalan’s polynomials, Byrd’s polynomials and also the  $k$ –Fibonacci numbers. Let  $h(x)$  be a polynomial with real coefficients. The  $h(x)$ –Fibonacci polynomials are defined by the recurrence relation

$$F_{h,n}(x) = h(x)F_{h,n-1}(x) + F_{h,n-2}(x), \quad n = 2, 3, \dots, \quad (1)$$

where  $F_{h,0}(x) = 0, F_{h,1}(x) = 1$ , for all polynomials  $h(x)$ . In [4], are studied basic properties of  $h(x)$ –Fibonacci polynomials. We note that due to essential results given in the paper [4] were obtained the results from the papers [2, 3], where were considered the  $x$ –Fibonacci polynomials.

Numerous generalizations of Fibonacci numbers generated different hypercomplex generalizations of Fibonacci numbers. Therefore, Fibonacci elements over some special algebras were intensively studied in the last time in various papers, as for example: [7] — [18]. All these papers studied properties of Fibonacci elements in complex numbers, or in quaternions and octonions, or in generalized Quaternion and Octonion algebras, or studied dual vectors or dual Fibonacci quaternions. At the same time, in the paper [19] were considered Fibonacci elements in an arbitrary finite-dimensional unitary algebra over a field  $K$  ( $K = \mathbb{R}, \mathbb{C}$ ) and were proved some basic properties of these hypercomplex numbers (generating functions, Binet formula, Cassini's identity, etc.).

In the paper [20] are defined the  $k$  – Fibonacci and the  $k$  – Lucas quaternions. For these quaternions were investigated the generating functions, Binet formula, formulae for some sums and Cassini's identity. Some of results of the paper [20] were generalized in the article [21], where was introduced the  $h(x)$  – Fibonacci quaternion polynomials which generalize the  $k$  – Fibonacci quaternion numbers. In [21], is presented a Binet-style formula, ordinary generating function and some basic identities for  $h(x)$  – Fibonacci quaternion polynomials.

In the paper [22] are defined  $h(x)$  – Fibonacci octonion polynomials. For the last mentioned, were obtained a similar Binet formula and generating function.

In this paper, we introduce  $h(x)$  – Fibonacci polynomials in an arbitrary finite dimensional unitary algebra over a field  $K$  ( $K = \mathbb{R}, \mathbb{C}$ ) which generalize both  $k$  – Fibonacci quaternions,  $h(x)$  – Fibonacci quaternion polynomials and  $h(x)$  – Fibonacci octonion polynomials. We also prove some relations between  $h(x)$  – Fibonacci polynomials in such an arbitrary algebra, Binet-style formula, Catalan-style identity, and generating function.

## Real $h(x)$ – Fibonacci polynomials and their properties

In this section we indicate some basic properties of  $h(x)$  – Fibonacci polynomials defined by the equality (1).

1. Generating function [4]:

$$g(t) = \frac{t}{1 - h(x)t - t^2}.$$

2. For  $n \in N$  (see [4]), we have

$$F_{h,n}(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \mathbb{C}_{n-k-1}^k h^{n-2k-1}(x). \quad (2)$$

3. For  $n \in N$  (see [4]), we have

$$F_{h,n}(x) = 2^{1-n} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \mathbb{C}_n^{2k+1} h^{n-2k-1}(x) (h^2(x) + 4)^k.$$

4. For  $n \in N$  (see [4]), we have

$$F_{h,n}(x) = i^{n-1} U_{n-1} \left( \frac{h(x)}{2i} \right),$$

where  $i^2 = -1$  and  $U_n(t) := \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^j \mathbb{C}_{n-j}^j (2t)^{n-2j}$  is the Chebyshev polynomial of the second kind.

5. Let  $\alpha(x)$  and  $\beta(x)$  denote the roots of the characteristic equation  $v^2 - h(x)v - 1 = 0$  of the recurrence relation (1). Then

$$\alpha(x) := \frac{h(x) + \sqrt{h^2(x) + 4}}{2}, \quad \beta(x) := \frac{h(x) - \sqrt{h^2(x) + 4}}{2}. \quad (3)$$

For all  $n = 0, 1, 2, \dots$  the Binet formula is of the form (see [4])

$$F_{h,n}(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}. \quad (4)$$

6. (see [3]):  $\lim_{n \rightarrow \infty} \frac{F_{h,n+1}(x)}{F_{h,n}(x)} = \alpha(x).$

7. (see [3]):

$$\sum_{k=1}^n F_{h,k}(x) = \frac{F_{h,n+1}(x) + F_{h,n}(x) - 1}{h(x)}. \quad (5)$$

8. For  $n, r$  integers and  $n > r$  we have the Catalan identity [3]:

$$F_{h,n-r}(x)F_{h,n+r}(x) - F_{h,n}^2(x) = (-1)^{n-r-1}F_{h,r}^2(x).$$

9. For  $a, b, c, d$  and  $r$  integers, with  $a + b = c + d$  we have [3]:

$$F_{h,a}(x)F_{h,b}(x) - F_{h,c}(x)F_{h,d}(x) = (-1)^r (F_{h,a-r}(x)F_{h,b-r}(x) - F_{h,c-r}(x)F_{h,d-r}(x)).$$

We note that the representation given in (2) can be rewritten in the following differential form

$$F_{h,n}(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{k!} \frac{d^k}{dh^k} h^{n-k-1}(x).$$

### $h(x)$ – Fibonacci polynomials in an arbitrary finite dimensional algebra

Let  $A$  be an unitary arbitrary  $(m + 1)$ -dimensional algebra over  $K$  ( $K = \mathbb{R}, \mathbb{C}$ ) with a basis  $\{e_0, e_1, e_2, \dots, e_m\}$ .

**Definition 3.1.** The  $h(x)$  – Fibonacci polynomials  $\{Q_{h,n}(x)\}_{n=0}^{\infty}$  in such an arbitrary algebra  $A$  are defined by the recurrent relation

$$Q_{h,n}(x) = \sum_{k=0}^m F_{h,n+k}(x) e_k, \quad (6)$$

where  $F_{h,n}(x)$  is the  $n$ th real  $h(x)$  – Fibonacci polynomial.

In the case where the algebra  $A$  coincides with the quaternion algebra  $\mathbb{H}$ , we obtain  $h(x)$  – Fibonacci quaternion polynomials, studied in [21]. If an algebra  $A$  coincides with the octonion algebra  $H$ , we obtain  $h(x)$  – Fibonacci octonion polynomials which were considered in [22].

**Proposition 3.2.** For any natural numbers  $n$  and  $p$  the following relations hold:

(i)

$$Q_{h,n+2}(x) = h(x)Q_{h,n+1}(x) + Q_{h,n}(x);$$

(ii)

$$\sum_{k=1}^p Q_{h,k}(x) = \frac{1}{h(x)} (Q_{h,p+1}(x) + Q_{h,p}(x) - Q_{h,0}(x) - Q_{h,1}(x)).$$

**Proof.** (i) directly follows from definition 3.1. (ii) In the following, instead of  $Q_{h,n}(x)$  and  $F_{h,j}(x)$  we will write  $Q_{h,n}$  and  $F_{h,j}$ , respectively. Using the identity (5), we have

$$\begin{aligned} \sum_{k=1}^p Q_{h,k} &= \sum_{j=0}^m F_{h,j+1} e_j + \sum_{j=0}^m F_{h,j+2} e_j + \dots + \sum_{j=0}^m F_{h,j+p} e_j = \\ &= e_0(F_{h,1} + F_{h,2} + \dots + F_{h,p}) + e_1(F_{h,2} + F_{h,3} + \dots + F_{h,p+1}) + \dots \\ &+ e_m(F_{h,m+1} + F_{h,m+2} + \dots + F_{h,m+p}) = \frac{e_0}{h(x)}(F_{h,p+1} + F_{h,p} - 1) + \\ &\frac{e_1}{h(x)}(F_{h,p+2} + F_{h,p+1} - 1 - h(x)F_{h,1}) + \dots \\ &+ \frac{e_m}{h(x)}(F_{h,p+m+1} + F_{h,p+m} - 1 - h(x)F_{h,1} - h(x)F_{h,2} - \dots - h(x)F_{h,m}). \end{aligned}$$

Since from (5) we have  $1 + h(x) \sum_{k=1}^r F_{h,k} = F_{h,r} + F_{h,r+1}$ , then we get

$$\begin{aligned} \sum_{k=1}^p Q_{h,k} &= \frac{e_0}{h(x)}(F_{h,p+1} + F_{h,p} - 1) + \frac{e_1}{h(x)}(F_{h,p+2} + F_{h,p+1} - F_{h,1} - F_{h,2}) + \cdots \\ &+ \frac{e_m}{h(x)}(F_{h,p+m+1} + F_{h,p+m} - F_{h,m} - F_{h,m+1}) = \frac{1}{h(x)}(Q_{h,p+1} + Q_{h,p} - Q_{h,0} - Q_{h,1}). \end{aligned}$$

The proposition is proved.

We obtain the following Binet formula for  $Q_{h,n}(x)$ .

**Theorem 3.3.** For  $n = 0, 1, 2, \dots$  we have the following relation

$$Q_{h,n}(x) = \frac{\alpha^*(x)\alpha^n(x) - \beta^*(x)\beta^n(x)}{\alpha(x) - \beta(x)}, \quad (7)$$

where  $\alpha^* = \sum_{k=0}^m \alpha^k(x) e_k$ ,  $\beta^* = \sum_{k=0}^m \beta^k(x) e_k$ .

**Proof.** Using the Binet-style formula (4), we obtain

$$\begin{aligned} Q_{h,n}(x) &= \sum_{k=0}^m F_{h,n+k}(x) e_k = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} e_0 + \frac{\alpha^{n+1}(x) - \beta^{n+1}(x)}{\alpha(x) - \beta(x)} e_1 + \cdots \\ &+ \frac{\alpha^{n+m}(x) - \beta^{n+m}(x)}{\alpha(x) - \beta(x)} e_m = \frac{\alpha^n(x)}{\alpha(x) - \beta(x)} (e_0 + \alpha(x)e_1 + \alpha^2(x)e_2 + \cdots \\ &+ \alpha^m(x)e_m) - \frac{\beta^n(x)}{\alpha(x) - \beta(x)} (e_0 + \beta(x)e_1 + \beta^2(x)e_2 + \cdots + \beta^m(x)e_m) = \\ &\frac{\alpha^*(x)\alpha^n(x) - \beta^*(x)\beta^n(x)}{\alpha(x) - \beta(x)}. \end{aligned}$$

**Remark 3.4.** The above result generalizes the Binet formulae from the papers [21] and [22].

**Definition 3.5.** The generating function  $G(t)$  of the sequence  $\{Q_{h,n}(x)\}_{n=0}^\infty$  is defined by

$$G(t) = \sum_{n=0}^\infty Q_{h,n}(x) t^n. \quad (8)$$

**Theorem 3.6.** The generating function for the  $h(x)$  - Fibonacci polynomials  $Q_{h,n}(x)$  in an arbitrary algebra is of the form

$$G(t) = \frac{Q_{h,0}(x) + (Q_{h,1}(x) - h(x)Q_{h,0}(x))t}{1 - h(x)t - t^2}.$$

**Proof.** Taking into account the equality (8), we consider the product

$$\begin{aligned} G(t)(1 - h(x)t - t^2) &= \sum_{n=0}^\infty Q_{h,n}(x) t^n - h(x) \sum_{n=0}^\infty Q_{h,n}(x) t^{n+1} - \sum_{n=0}^\infty Q_{h,n}(x) t^{n+2} = \\ &Q_{h,0}(x) + (Q_{h,1}(x) - h(x)Q_{h,0}(x))t + \sum_{n=2}^\infty t^n (Q_{h,n} - h(x)Q_{h,n-1} - Q_{h,n-2}) = \\ &Q_{h,0}(x) + (Q_{h,1}(x) - h(x)Q_{h,0}(x)). \end{aligned}$$

The theorem is proved.

**Remark 3.7.** The above Theorem generalizes results from the papers [21] and [22].

**Theorem 3.8.** (Catalan's identity) For nonnegative integer numbers  $n, r$ , such that  $r \leq n$ , we have

$$Q_{h,n+r}(x)Q_{h,n-r}(x) - Q_{h,n}^2(x) =$$

$$\frac{(-1)^{n+r+1}}{h^2(x) + 4}(\alpha^*(x)\beta^*(x)[(-1)^{r+1} + \alpha^2(x)] + \beta^*(x)\alpha^*(x)[(-1)^{r+1} + \beta^2(x)]).$$

**Proof.** Using the formula (7), we have

$$\begin{aligned} Q_{h,n+r}(x)Q_{h,n-r}(x) - Q_{h,n}^2(x) = \\ \frac{1}{(\alpha(x) - \beta(x))^2}(\alpha^*(x)\beta^*(x)(\alpha(x)\beta(x))^n \left[1 - \left(\frac{\alpha(x)}{\beta(x)}\right)^r\right] + \\ \beta^*(x)\alpha^*(x)(\alpha(x)\beta(x))^n \left[1 - \left(\frac{\beta(x)}{\alpha(x)}\right)^r\right]). \end{aligned}$$

Now, taking into account the relations  $\alpha(x)\beta(x) = -1$ ,  $\frac{\alpha(x)}{\beta(x)} = -\alpha^2(x)$ ,  $\frac{\beta(x)}{\alpha(x)} = -\beta^2(x)$ , we obtain the statement of the theorem. The theorem is now proved.

If in the Theorem 3.8 we set  $r = 1$ , we obtain the Cassini-style identity.

**Corollary 3.9.** (Cassini's identity) *For any natural number  $n$ , we have*

$$\begin{aligned} Q_{h,n+1}(x)Q_{h,n-1}(x) - Q_{h,n}^2(x) = \\ \frac{(-1)^n}{h^2(x) + 4}(\alpha^*(x)\beta^*(x)[1 + \alpha^2(x)] + \beta^*(x)\alpha^*(x)[1 + \beta^2(x)]). \end{aligned}$$

**Remark 3.10.** Theorem 3.8 and Corollary 3.9 generalize the Theorems 3.8 and 3.9, respectively, from the paper [21].

Similarly to proof of Theorem 3.8, can be proved the following result.

**Theorem 3.11.** (d'Ocagne's identity) *Suppose that  $n$  is a nonnegative integer number and  $r$  is any natural number. Then for  $r > n$ , we have*

$$\begin{aligned} Q_{h,r}(x)Q_{h,n+1}(x) - Q_{h,r+1}(x)Q_{h,n}(x) = \\ \frac{(-1)^n}{\alpha(x) - \beta(x)}(\alpha^*(x)\beta^*(x)\alpha^{r-n}(x) - \beta^*(x)\alpha^*(x)\beta^{r-n}(x)). \end{aligned}$$

**Acknowledgement.** The second author is partially supported by Grant of Ministry of Education and Science of Ukraine (Project No. 0116U001528).

## References

- [1] Falcón S, Plaza Á. On the Fibonacci  $k$ -numbers. Chaos, Solitons and Fractals 2007;32(5):1615–24.
- [2] S. Falcón and A. Plaza, The  $k$ -Fibonacci sequence and the Pascal 2-triangle. Chaos, Solitons and Fractals 2007;33(1):38–49.
- [3] S. Falcón and A. Plaza, On  $k$ -Fibonacci sequences and polynomials and their derivatives. Chaos, Solitons and Fractals 39(3)(2009), 1005–1019.
- [4] Nalli A, Haukkanen P. On generalized Fibonacci and Lucas polynomials. Chaos, Solitons and Fractals 2009;42(5):3179–86.
- [5] Akyiğit M., Köksal H. H., Tosun M. Split Fibonacci Quaternions. Adv. Appl. Clifford Algebras 23 no. 3, 535–545 (2013)
- [6] M. Akyigit, H. H. Koksall and M. Tosun, Fibonacci Generalized Quaternions. Adv. Appl. Clifford Algebras, 24, no. 3, 631–641 (2014)
- [7] I. Akkus, O. Keçilioğlu, Split Fibonacci and Lucas Octonions, Adv. Appl. Clifford Algebras, 25, no. 3, 517–525 (2015)

- [8] O. Keçilioğlu, I. Akkus, The Fibonacci Octonions, *Adv. Appl. Clifford Algebras*, 25(1)(2015), 151–158.
- [9] C. Flaut, D. Savin, Quaternion Algebras and Generalized Fibonacci-Lucas Quaternions, *Adv. Appl. Clifford Algebras*, 25, no. 4, 853–862 (2015)
- [10] C. Flaut, V. Shpakivskyi, Real matrix representations for the complex quaternions, *Adv. Appl. Clifford Algebras*, 23(3)(2013), 657–671.
- [11] C. Flaut, V. Shpakivskyi, On Generalized Fibonacci Quaternions and Fibonacci-Narayana Quaternions, *Adv. Appl. Clifford Algebras*, 23(3)(2013), 673–688.
- [12] I. A. Guren, S.K. Nurkan, A new approach to Fibonacci, Lucas numbers and dual vectors, *Adv. Appl. Clifford Algebras*, 25, no. 3, 577–590 (2015)
- [13] S. K. Nurkan, I. A. Guren, Dual Fibonacci quaternions, *Adv. Appl. Clifford Algebras*, 25, no. 2, 403–414 (2015)
- [14] S. Halici, On Fibonacci Quaternions, *Adv. in Appl. Clifford Algebras*, 22(2)(2012), 321–327.
- [15] S. Halici, On Complex Fibonacci Quaternions. *Adv. Appl. Clifford Algebras*. 23(1), 105–112 (2013)
- [16] S. Yüce, F.T. Aydın, A New Aspect of Dual Fibonacci Quaternions, *Adv. Appl. Clifford Algebras*, 26(2)(2016), 873–884.
- [17] A. F. Horadam, Complex Fibonacci Numbers and Fibonacci Quaternions, *Amer. Math. Monthly*, 70(1963), 289–291.
- [18] M. N. S. Swamy, On generalized Fibonacci Quaternions, *The Fibonacci Quarterly*, 11(5)(1973), 547–549.
- [19] C. Flaut, V. Shpakivskyi, Some remarks about Fibonacci elements in an arbitrary algebra, *Bull. Soc. Sci. Lettres Łódź*, 65 (3), 63–73 (2015).
- [20] J. L. Ramirez, Some Combinatorial Properties of the  $k$ -Fibonacci and the  $k$ -Lucas Quaternions, *An. St. Univ. Ovidius Constanta, Mat. Ser.*, 23 (2)(2015), 201–212
- [21] Catarino, P. A note on  $h(x)$  – fibonacci quaternion polynomials. *Chaos, Solitons and Fractals*, 77 (2015), 1–5.
- [22] İpec A., Arı K. On  $h(x)$  – fibonacci octonion polynomials. *Alabama J. Math.*, 39 (2015), 1–6.

Cristina Flaut  
 Faculty of Mathematics and Computer Science,  
 Ovidius University,  
 Bd. Mamaia 124, 900527, Constanta,  
 Romania  
<http://cristinaflaut.wikispaces.com/>  
<http://www.univ-ovidius.ro/math/>  
 e-mail: cflaut@univ-ovidius.ro, cristina \_ flaut@yahoo.com

Vitalii Shpakivskyi  
 Department of Complex Analysis and Potential Theory,  
 Institute of Mathematics of the National Academy of Sciences of Ukraine,  
 3, Tereshchenkivs'ka st.  
 01601 Kyiv-4,  
 Ukraine  
<http://www.imath.kiev.ua/>  
 e-mail: shpakivskyi86@gmail.com, shpakivskyi@imath.kiev.ua

Elena Vlad  
 National College "Emil Botta", Adjud, Vrancea, Romania  
[elena\\_brinzo2000@yahoo.com](mailto:elena_brinzo2000@yahoo.com)